

Curvature and torsion of quantum evolution

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Abstract

We study characteristics of quantum evolution which can be called curvature and torsion. The curvature shows a deviation of the state vector of the quantum evolution from the geodesic line and the torsion shows a deviation of state vector from the plane of evolution (a two-dimensional subspace) at a given time.

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1 Introduction

Geometric ideas play an important role in quantum mechanics, in particular in the studies of the quantum evolution [1, 2, 3], quantum brachistochrone problem [4, 5], entanglement of quantum states [6, 7], Berry phase has geometric origin [8].

In the classical case the curvature and torsion are important geometric characteristics of the trajectory. The aim of the present paper is to answer

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the question: What is the quantum analogue of these classical geometrical notions? Partly the answer on this question was given in [9] where the authors from a different perspective than in this paper, namely, considering geometry of quantum statistical interference, derived explicit expression for the curvature of quantum evolution.

First let us consider some facts of the geometry of the space of quantum states. A distance between two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ can be defined in different ways. In this paper we will refer to the Fubini-Study distance and the Wootters distance defined, respectively, as follows (for a short review see, for instance, [10])

$$d^{(\text{FS})}(|\psi_1\rangle, |\psi_2\rangle) = \gamma \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}, \quad (1)$$

$$d^{(\text{W})}(|\psi_1\rangle, |\psi_2\rangle) = \gamma \arccos |\langle\psi_1|\psi_2\rangle|, \quad (2)$$

where γ is an arbitrary constant.

These distances are equivalent for the neighboring states when $|\langle\psi_1|\psi_2\rangle|^2 = 1 - \delta^2$, where δ is a small value, namely, $d^{(\text{FS})} = d^{(\text{W})} = \gamma\delta$. Likewise we have for other definitions of distance. As a result of this the element of length for the family (set) of quantum state vectors $|\psi(\xi^1, \xi^2, \dots, \xi^k)\rangle$ parametrized by k parameters $\xi^1, \xi^2, \dots, \xi^k$ is the same for different definitions of distance

$$ds^2 = g_{ij} d\xi^i d\xi^j \quad (3)$$

with metric tensor

$$g_{ij} = \gamma^2 \text{Re}(\langle\psi_i|\psi_j\rangle - \langle\psi_i|\psi\rangle\langle\psi|\psi_j\rangle), \quad (4)$$

where

$$|\psi_i\rangle = \frac{\partial}{\partial \xi^i} |\psi(\xi^1, \xi^2, \dots, \xi^k)\rangle. \quad (5)$$

This form of metrics for quantum states was discussed by many authors [2, 11, 12, 13, 14].

It is convenient to put $\gamma = 2$. Then in a two-dimensional case g_{ij} is metric tensor of a sphere with the radius equal one (the Bloch sphere).

Considering evolution of a quantum state according to the Schrödinger equation one can introduce the velocity of quantum evolution [1]

$$v = \frac{ds}{dt} = \gamma \sqrt{\langle(\Delta H)^2\rangle}/\hbar, \quad (6)$$

where $\Delta H = H - \langle H \rangle$.

2 Geodesic in the space of quantum state vectors

The geodesic line (one-parametric set of the quantum state vectors) that connects two state vectors $|\psi_0\rangle$ and $|\psi_1\rangle$ can be defined as their linear combination

$$|\psi(\xi)\rangle = C[(1 - \xi)|\psi_0\rangle + \xi|\psi_1\rangle e^{i\phi}], \quad (7)$$

where ξ is a real parameter changing from 0 to 1. We choose the phase multiplier $e^{i\phi}$ from the following condition. Note that $|\psi_0\rangle$ and $e^{i\phi_0}|\psi_0\rangle$ describe the same quantum state, similarly, $|\psi_1\rangle$ and $e^{i\phi_1}|\psi_1\rangle$ describe the same quantum state. Therefore, we require that geodesic lines defined between the states $|\psi_0\rangle$, $|\psi_1\rangle$ and between the states $e^{i\phi_0}|\psi_0\rangle$, $e^{i\phi_1}|\psi_1\rangle$ coincide. This requirement is satisfied if we choose

$$e^{i\phi} = \frac{\langle\psi_1|\psi_0\rangle}{|\langle\psi_1|\psi_0\rangle|}. \quad (8)$$

The normalization condition $\langle\psi(\xi)|\psi(\xi)\rangle = 1$ gives

$$C = \frac{1}{\sqrt{1 - 2\xi(1 - \xi)(1 - |\langle\psi_1|\psi_0\rangle|)}}. \quad (9)$$

The geodesic line (7) is a one-parametric set of states and there exist many possibilities to parametrize it. One can show that the length of the curve in quantum space does not depend on the way of its parametrization. For the calculation of the length of the geodesic line it is convenient to write its equation as follows

$$|\psi(\theta)\rangle = C[\sin(\theta/2)|\psi_0\rangle + \cos(\theta/2)|\psi_1\rangle e^{i\phi}], \quad (10)$$

here new parameter $0 \leq \theta \leq \pi$ and normalization constant $C = 1/\sqrt{1 + |\langle\psi_1|\psi_0\rangle| \sin \theta}$. Stress once more that (7) and (10) describe the same one-parametric family of quantum state vectors, namely, the geodesic line.

Using (4) for the one-parameter set of states (10) we obtain

$$ds = \frac{\gamma}{2} \frac{\sqrt{1 - |\langle\psi_1|\psi_0\rangle|^2}}{1 + |\langle\psi_1|\psi_0\rangle| \sin \theta} d\theta. \quad (11)$$

Then the length of the geodesic line connecting the states $|\psi_0\rangle$ and $|\psi_1\rangle$ is

$$s = \int ds = \gamma \arccos |\langle \psi_1 | \psi_0 \rangle|. \quad (12)$$

As it should be, the length of the geodesic line is equal to the Wootters distance between the corresponding state vectors.

In conclusion of this section let us note that we can calculate the length of curve (7) connecting the states $|\psi_0\rangle$ and $|\psi_1\rangle$ for an arbitrary phase ϕ . Then the geodesic line can be defined as the one having minimal length. One can find that minimal length is achieved for ϕ given in (8) and is equal to the Wootters distance.

3 Curvature

The state vector of the quantum evolution depends on one parameter, namely, on time t and is one-parametric set of state vectors $|\psi(t)\rangle = \exp(-iHt)|\psi_0\rangle$ generated by the Hamiltonian of the system. The deviation of evolution state vector $|\psi(t)\rangle$ from the geodesic connecting the same two state vectors is related with the curvature of quantum evolution.

In order to introduce the curvature as well as the torsion we consider the evolution in two stages. At first, we consider the evolution during the time Δt from an initial state $|\psi_0\rangle$ to

$$|\psi'\rangle = e^{-iH\Delta t/\hbar}|\psi_0\rangle \quad (13)$$

and then – during $\Delta t'$ from $|\psi'\rangle$ to

$$|\psi_1\rangle = e^{-iH\Delta t'/\hbar}|\psi'\rangle = e^{-iH(\Delta t+\Delta t')/\hbar}|\psi_0\rangle. \quad (14)$$

where H is a time independent Hamiltonian. In this section without the loss of generality we put $\Delta t = \Delta t'$.

A deviation of the quantum evolution from the geodesic line connecting $|\psi_0\rangle$ and $|\psi_1\rangle$ can be characterized by the maximum value of $|\langle \psi' | \psi(\xi) \rangle|^2$ with respect to ξ . When $\max |\langle \psi' | \psi(\xi) \rangle|^2 = 1$ then the state $|\psi'\rangle$ belongs to the geodesic. A deviation of $|\psi'\rangle$ from the geodesic is larger when $\max |\langle \psi' | \psi(\xi) \rangle|^2$ is smaller. It is convenient to introduce the following expression $1 - \max |\langle \psi' | \psi(\xi) \rangle|^2 = \min(1 - |\langle \psi' | \psi(\xi) \rangle|^2)$ which equals zero when the deviation is zero and is positive otherwise becoming larger as the deviation is larger. This expression

multiplied by some constant γ^2 is just a squared Fubini-Study distance. So, a deviation of the quantum evolution from the geodesic line connecting $|\psi_0\rangle$ and $|\psi_1\rangle$ can be characterized by the distance between the state $|\psi'\rangle$ and the geodesic line $|\psi(\xi)\rangle$

$$d^2 = \min d^2(\xi) = \min \gamma^2(1 - |\langle\psi'|\psi(\xi)\rangle|^2). \quad (15)$$

The minimal value of this expression is achieved at $\xi = 1/2$. Taking into account the terms of order $(\Delta t)^4$ we find

$$d^2 = \frac{\gamma^2}{4} (\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2) \frac{(\Delta t)^4}{\hbar^4} = \frac{\gamma^2}{4} \kappa \frac{(\Delta t)^4}{\hbar^4}. \quad (16)$$

Here, the multiplier

$$\kappa = \langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2 \quad (17)$$

can be called the curvature coefficient or, simply, curvature. It is convenient to introduce a dimensionless curvature coefficient

$$\bar{\kappa} = \frac{\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2}{\langle(\Delta H)^2\rangle^2}. \quad (18)$$

For the first time this result was obtained in [9] in the frame of study of the geometry of quantum statistical interference.

Now we show that the curvature of the quantum evolution can also be obtained using the geometric treatment. For small time the classical motion along a given curve can be treated as a motion along a circle with radius R for which we can write

$$\frac{1}{R} = \frac{2d}{(s/2)^2}, \quad (19)$$

where s is the length of the curve between two neighboring points on it, which can be considered as an arc of the circle, and d is the distance between the middle point of an arc and the chord connecting these two points. Similarly to (19) we now define the radius of curvature for the quantum evolution. In our case d is given by (16) and

$$s = v2\Delta t = \gamma \frac{\sqrt{\langle(\Delta H)^2\rangle}}{\hbar} 2\Delta t \quad (20)$$

is the length that a quantum system passes during the time $2\Delta t$ of the evolution. Here v is the velocity of quantum evolution given in (6). As a result we have

$$\frac{1}{R^2} = \frac{1}{\gamma^2} \frac{\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2}{\langle(\Delta H)^2\rangle^2} = \frac{\bar{\kappa}}{\gamma^2}. \quad (21)$$

4 Torsion

Torsion is related with the deviation of evolution state vector from the plane of evolution (a two-dimensional subspace) at a given time.

In order to find torsion we consider the evolution in the two stages given by (13) and (14). Two vectors $|\psi_0\rangle$ and $|\psi'\rangle$ form the first stage define the plane of evolution. Using these vectors we can construct the orthogonal ones

$$|\phi_1\rangle = \frac{1}{\sqrt{2(1+a)}} \langle(|\psi_0\rangle + e^{-i\alpha}|\psi'\rangle), \quad (22)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2(1+a)}} \langle(|\psi_0\rangle - e^{-i\alpha}|\psi'\rangle), \quad (23)$$

where a and ϕ are defined by $\langle\psi_0|\psi'\rangle = ae^{i\alpha}$. Then the unit operator in a two-dimensional subspace spanned by $|\phi_1\rangle$ and $|\phi_2\rangle$ is

$$\hat{I}_2 = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|. \quad (24)$$

Note that it is the projection operator of an arbitrary state vector on a two-dimensional subspace.

In order to find the deviation of the state vector $|\psi_1\rangle$ obtained on the second stage from the plane of evolution we calculate the mean value of \hat{I}_2

$$I_2 = \langle\psi_1|\hat{I}_2|\psi_1\rangle = |\langle\phi_1|\psi_1\rangle|^2 + |\langle\phi_2|\psi_1\rangle|^2. \quad (25)$$

When $I_2 = 1$ then $|\psi_1\rangle$ belongs to the subspace spanned by $|\phi_1\rangle$ and $|\phi_2\rangle$. It means that three state vectors $|\psi_0\rangle$, $|\psi'\rangle$, $|\psi_1\rangle$ belong to the same plane (the two-dimensional subspace) of evolution and thus the torsion is zero. The expression $1 - I_2$ gives the magnitude of the torsion. Considering small Δt and $\Delta t'$ and taking into account the terms up to fourth order we find

$$1 - I_2 = \left(\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2 - \frac{\langle(\Delta H)^3\rangle^2}{\langle(\Delta H)^2\rangle} \right) \frac{\Delta t^2(\Delta t + \Delta t')^2}{4\hbar^4}. \quad (26)$$

The coefficient

$$\tau = \langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2 - \frac{\langle(\Delta H)^3\rangle^2}{\langle(\Delta H)^2\rangle} \quad (27)$$

does not depend on Δt and $\Delta t'$ and can be called the torsion coefficient. For simplicity we put $\Delta t = \Delta t'$ and then

$$1 - I_2 = \tau \frac{\Delta t^4}{\hbar^4}. \quad (28)$$

Similarly to the dimensionless curvature coefficient we introduce a dimensionless torsion coefficient

$$\bar{\tau} = \frac{\tau}{\langle(\Delta H)^2\rangle^2} = \frac{\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2}{\langle(\Delta H)^2\rangle^2} - \frac{\langle(\Delta H)^3\rangle^2}{\langle(\Delta H)^2\rangle^3} = \bar{\kappa} - \frac{\langle(\Delta H)^3\rangle^2}{\langle(\Delta H)^2\rangle^3}. \quad (29)$$

Now let us show that $1 - I_2$ has a geometrical meaning, namely, it is proportional to the squared distance of the state $|\psi_1\rangle$ to the plane of quantum evolution spanned by $|\phi_1\rangle$ and $|\phi_2\rangle$. The distance between a given state $|\psi_1\rangle$ and the plane is that between $|\psi_1\rangle$ and the normalized projection of this vector onto the plane. The normalized projection of $|\psi_1\rangle$ on the plane is

$$|\tilde{\psi}_1\rangle = c \hat{I}_2 |\psi_1\rangle, \quad (30)$$

where from the condition $\langle\tilde{\psi}_1|\tilde{\psi}_1\rangle = 1$ we find $c = 1/\sqrt{\langle\psi_1|\hat{I}_2\hat{I}_2|\psi_1\rangle} = 1/\sqrt{\langle\psi_1|\hat{I}_2|\psi_1\rangle}$. Here we use that $(\hat{I}_2)^2 = \hat{I}_2$. Then the squared distance between the state $|\psi_1\rangle$ and the plane is

$$d^2 = \gamma^2(1 - |\langle\psi_1|\tilde{\psi}_1\rangle|^2) = \gamma^2(1 - |\langle\psi_1|\hat{I}_2|\psi_1\rangle|) = \gamma^2(1 - I_2), \quad (31)$$

where we use that $|I_2| = I_2$.

5 Discussion

In this paper we have obtained the curvature and torsion coefficients (18) and (29) for the quantum evolution which is governed by a time independent Hamiltonian. In this case the curvature and torsion coefficients are constant.

The evolution is going along the geodesic when $\bar{\kappa} = 0$ or explicitly

$$\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2 = 0. \quad (32)$$

Introducing operator $\hat{A} = (\Delta H)^2$ we rewrite this condition in the form $\langle (\Delta \hat{A})^2 \rangle = 0$. Then we find that (32) is equivalent to the equation $\Delta \hat{A}|\psi\rangle = 0$ which explicitly reads

$$(\Delta H)^2|\psi\rangle = \langle (\Delta H)^2 \rangle |\psi\rangle. \quad (33)$$

The solution of this equation is

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|E_1\rangle + e^{i\alpha}|E_2\rangle), \quad (34)$$

where α is an arbitrary phase, $|E_1\rangle$ and $|E_2\rangle$ are two eigenstates of the Hamiltonian H with eigenenergies E_1 and E_2 . Considering (34) as an initial state for time dependent state that evolves along the geodesic we find

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} (e^{-iE_1t/\hbar}|E_1\rangle + e^{i\alpha}e^{-iE_2t/\hbar}|E_2\rangle). \quad (35)$$

One can find that for an arbitrary fixed time the evolution state vector (35) satisfies equation (33) and the curvature for this evolution is zero. It is worth stressing that the state vector of the geodesic evolution contains only two eigenstates of the Hamiltonian and lies in a two-dimensional subspace.

Now let us show that the torsion of the geodesic is zero. Using (33) we have

$$\langle (\Delta H)^3 \rangle = \langle \psi | \Delta H (\Delta H)^2 | \psi \rangle = \langle (\Delta H)^2 \rangle \langle \psi | \Delta H | \psi \rangle = 0. \quad (36)$$

Then according to (29) and remembering that for the geodesic $\bar{\kappa} = 0$ we find that the torsion $\bar{\tau} = 0$.

Let us verify that for two-dimensional space the torsion given by (29) is zero as this is abiding by the definition. The most general Hamiltonian for a two-dimensional case reads

$$H = \omega(\boldsymbol{\sigma}\mathbf{n}) + \epsilon, \quad (37)$$

where \mathbf{n} is a unit vector. Note that the curvature and torsion depend on ΔH , where ϵ is canceled. So, with out loss the generality we put $\epsilon = 0$. Then using the properties of Pauli matrices we find for Hamiltonian (37) with $\epsilon = 0$ the following results $\langle (\Delta H)^2 \rangle = \omega^2 - \langle H \rangle^2$, $\langle (\Delta H)^4 \rangle - \langle (\Delta H)^2 \rangle^2 = 4\langle H \rangle^2 \langle (\Delta H)^2 \rangle$ and $\langle (\Delta H)^3 \rangle = -2\langle H \rangle \langle (\Delta H)^2 \rangle$. Then one can find that the torsion (29) in

the two-dimensional case is always zero. For curvature in this case we have $\bar{\kappa} = 4\langle H \rangle^2 / \langle (\Delta H)^2 \rangle$. Thus in two-dimensional case the quantum evolution is going along the geodesic line when $\langle H \rangle = \omega \langle (\boldsymbol{\sigma} \mathbf{n}) \rangle = 0$.

In conclusion let us note an interesting fact which follows from (29). Namely, for symmetric states when $\langle (\Delta H)^3 \rangle = 0$ we find that $\bar{\kappa} = \bar{\tau}$. It means that the curvature and torsion during the evolution of symmetric states are strongly related.

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